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Müntz's Theorem for Group Algebras

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I. INTRODUCTION

It is natural to attempt to extend results from the theory of approximation in the Banach algebra $C[X]$ (the algebra of continuous complex-valued functions on the compact Hausdorff space X) to other semi-simple commutative Banach algebras. For example, Katznelson and Rudin [5] have studied the possibility of extending the Stone–Weierstrass theorem. This paper considers the possibility of extending Müntz's theorem to some semi-simple commutative Banach algebras; in particular, we obtain some results for certain group algebras. The crux of the difficulty in extending approximation theory results for $C[X]$ to the more general situation lies in the fact that the Gelfand transform is norm-shrinking.

Recall that the classical Müntz theorem on $C[0, 1]$ is (Davis [2], p. 272):

THEOREM 1.1 (Müntz). *If $\{\mu_n\}_1^\infty$ is a strictly increasing sequence of positive numbers, $\mu_n \to \infty$, then $\{1, x^{\mu_1}, x^{\mu_2},...\}$ is complete in $C[0, 1]$ if and only if $\sum 1/\mu_n = \infty$.*

The Müntz-type problem we consider is the following:

Let A be a semi-simple Banach algebra and let $f \in A$; let B be the closed subalgebra of A generated by f and let $\{\mu_n\}_1^\infty$ be an infinite sequence of distinct positive integers (or distinct positive numbers, $\mu_n \not\to 0$). Find sufficient conditions on $\{\mu_n\}_1^\infty$ and/or f in order that $\{f^{\mu_n}\}_1^\infty$ is complete in B.

Note that the above problem is phrased so that $\{f, f^2, f^3,...\}$ is complete in B by default, and hence the problem is formally independent of whether or not B has the "Stone–Weierstrass property"—the question considered by Katznelson and Rudin.

In Section II we give a solution to this problem for the case when B is an algebra which is generated by its idempotents whose Gelfand transforms have finite support. As a corollary we obtain a Müntz theorem on certain closed subalgebras of $A(\Gamma)$ where Γ is a discrete locally compact abelian

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group. Also we mention what can happen for some closed subalgebras of $A(Z)$ which are not spanned by their idempotents. In Section III we give a solution to the problem for the case $A = A(T)$, the algebra of absolutely convergent Fourier series. The main result is Theorem 3.14.

In order to clarify the terminology: Z is the group of integers; T is the circle group; Γ designates a locally compact abelian group whose dual is G. $A(\Gamma)$ is the Banach algebra consisting of the Fourier transforms of the elements of $L^1(G)$; multiplication in $A(\Gamma)$ is "pointwise" and the norm for elements of $A(\Gamma)$ is

$$\|f\|_{A(\Gamma)} = \|g\|_{L^1(G)},$$

where f is the Fourier transform of g. The Gelfand transform is designated by "^".

II. Müntz's Theorem for $A(\Gamma)$, Γ Discrete

The following result of Newman, Passow, and Raymon [9] gives a hint of the type of Müntz theorem we can expect for $A(\Gamma)$.

Theorem 2.1. *Let $X = \{0, x_n\}_1^\infty$ be a sequence of points in* $[0, 1]$ *such that $x_n \downarrow 0$. Let $\{\mu_n\}_1^\infty$ be any infinite sequence of distinct positive numbers, $\mu_n \not\to 0$. Then $\{1, x^{\mu_n}\}_1^\infty$ is complete in $C[X]$.*

Note the absence of the condition "$\sum (1/\mu_n) = \infty$" which appears in Theorem 1.1. It might seem that this is possible due to the fact that X is discrete and countable. Hence it is plausible to anticipate a similar result if we replace $C[X]$ by a commutative Banach algebra A whose maximal ideal space is discrete and countable. The next theorem makes this more precise.

Theorem 2.2. *Suppose A is a semi-simple commutative Banach algebra which is spanned by its idempotents whose Gelfand transforms have finite support in $\Delta(A)$, the maximal ideal space of A. Suppose $f \in A$ and* (a) *$|\hat{f}(x_1)| = |\hat{f}(x_2)|$ only if $x_1 = x_2$* for *$x_1, x_2 \in \Delta(A)$*; (b) *$\{|\hat{f}(x)| > \epsilon \mid x \in \Delta(A)\}$ is finite for every $\epsilon > 0$*; (c) *$\hat{f}(x) \neq 0$ for any $x \in \Delta(A)$. Then $\{f^{\mu_n}\}_1^\infty$ is complete in A for every infinite sequence $\{\mu_n\}_1^\infty$ of distinct positive integers.*

Proof. Let M be the closed span of $\{f^{\mu_n}\}_{n=1}^\infty$ in A. It suffices to show that every idempotent of A whose Gelfand transform has finite support is in M.

First we show, given $x \in \Delta(A)$ there is an idempotent $\gamma \in A$ such that

$$\begin{aligned} \hat{\gamma}(x) &= 1 \\ \hat{\gamma}(y) &= 0 \qquad \text{for} \quad y \in \Delta(A), \quad y \neq x. \end{aligned} \tag{2.2a}$$

Clearly, since $|\hat{f}(x)| > 0$ for every $x \in \Delta(A)$ and since A is spanned by its idempotents with finite support, the set

$$G = \{g \in A \mid g^2 = g, \hat{g} \text{ has finite support}, \hat{g}(x) = 1\}$$

is not empty. If $g_1, g_2 \in G$ then clearly $g_1 g_2 \in G$ and

$$\operatorname{supp}(g_1 g_2)\hat{} \subset (\operatorname{supp} \hat{g}_1) \cap (\operatorname{supp} \hat{g}_2).$$

Thus G contains an element, γ, with minimal support, i.e., $\operatorname{supp} \hat{\gamma} \subset \operatorname{supp} \hat{g}$ for every $g \in G$. Suppose $y \in \operatorname{supp} \hat{\gamma}$ and $y \neq x$; we show that this is a contradiction: since the idempotents with finite support span A and since $|\hat{f}|$ separates points on $\Delta(A)$, there is an idempotent $g_y \in A$ such that $\hat{g}_y$ has finite support, and such that $x \notin \operatorname{supp} \hat{g}_y$ and $y \in \operatorname{supp} \hat{g}_y$. Now let $\gamma_1 = \gamma - \gamma g_y$. Then $\hat{\gamma}_1$ has finite support, $\gamma_1^2 = \gamma_1$ and $\hat{\gamma}_1(x) = \hat{\gamma}(x) - \hat{\gamma}(x)\,\hat{g}_y(x) = 1 - 1 \cdot 0 = 1$. Thus $\gamma_1 \in G$. Moreover, $\operatorname{supp} \hat{\gamma}_1 \subset \operatorname{supp} \hat{\gamma}$ since in particular $\hat{\gamma}_1(y) = \hat{\gamma}(y) - \hat{\gamma}(y)\,\hat{g}(y) = 1 - 1 \cdot 1 = 0$. Thus γ does not have minimal support; contradiction.

To show that every idempotent with finite support is in M, it suffices to show that every idempotent of the form (2.2a) is in M.

The elements of $\Delta(A)$ can be ordered $x_1, x_2, x_3, \ldots$ such that $|\hat{f}(x_1)| > |\hat{f}(x_2)| > \cdots$. Thus there is a corresponding order $\gamma_1, \gamma_2, \gamma_3, \ldots$ for every idempotent of the form (2.2a):

$$\hat{\gamma}_j(x_j) = 1$$
$$\hat{\gamma}_j(x_k) = 0 \qquad \text{for all} \quad k \neq j, \quad j = 1, 2, 3, \ldots.$$

Proceed by induction; suppose we have shown $\gamma_j \in M$ for $j = 1, 2, \ldots, k$. Claim $\gamma_{k+1} \in M$: Let

$$f_1 = \frac{1}{\hat{f}(x_{k+1})}\left[f - \sum_{j=1}^{k+1} \hat{f}(x_j)\,\gamma_j\right].$$

Then $f_1 \in A$ and clearly

$$\|\hat{f}_1\|_\infty = \left|\frac{\hat{f}(x_{k+2})}{\hat{f}(x_{k+1})}\right| = r < 1.$$

Also,

$$f_1^{\mu_n} = \frac{1}{\hat{f}(x_{k+1})^{\mu_n}}\left[f^{\mu_n} - \sum_{j=1}^{k+1} \hat{f}(x_j)^{\mu_n}\,\gamma_j\right]$$
$$= \frac{1}{\hat{f}(x_{k+1})^{\mu_n}}\left[\left(f^{\mu_n} - \sum_{j=1}^{k} \hat{f}(x_j)^{\mu_n}\,\gamma_j\right)\right] - \gamma_{k+1}. \tag{2.2b}$$

But $\lim \|f_1^{\mu_n}\|^{1/\mu_n} = r$ by the spectral radius theorem. So if $\epsilon > 0$ and $r + \epsilon < 1$, we have for large enough n,

$$\|f_1^{\mu_n}\| \leqslant (r + \epsilon)^{\mu_n}.$$

Thus $\|f_1^{\mu_n}\| \to 0$ as $n \to \infty$. Thus from (2.2b) and the induction hypothesis we must have $\gamma_{k+1} \in M$. The initial step of the induction procedure is vacuous. Thus $\gamma_j \in M, j = 1, 2, 3,...$ and the proof is complete.

Obviously Theorem 2.1 is a special case of Theorem 2.2. As another special case of Theorem 2.2 we obtain our Müntz theorem for subalgebras of $A(\Gamma)$, Γ discrete, as Corollary 2.4 below. First we state the following definition (see Kahane [3]).

DEFINITION 2.3. Let B be a subalgebra of the commutative Banach algebra A. Define a relation, $\sim$, on $\Delta(A)$ by $x_1 \sim x_2$ if $\hat{g}(x_1) = \hat{g}(x_2)$ for every $g \in B$. This is an equivalence relation and partitions $\Delta(A)$ into equivalence classes $\{E_\alpha\}$ called the Rudin equivalence classes. $E_0 = \{x \in \Delta(A) \mid g(x) = 0$ for all $g \in A\}$ is called the "zero Rudin equivalence class." All others are "nonzero Rudin equivalence classes."

COROLLARY 2.4. *Suppose Γ is a discrete locally compact abelian group and B is a closed subalgebra of $A(\Gamma)$ which is spanned by its idempotents. Suppose $f \in B$ and*

(a) *For $x_1, x_2 \in \Gamma$,*

$$|f(x_1)| = |f(x_2)| \text{ only if } g(x_1) = g(x_2) \text{ for every } g \in B.$$

(b) *For $x \in \Gamma$,*

$$f(x) = 0 \text{ only if } g(x) = 0 \text{ for every } g \in B.$$

Then $\{f^{\mu_n}\}_1^\infty$ is complete in B for every infinite sequence $\{\mu_n\}_1^\infty$ of distinct positive integers.

Proof. The corollary is an immediate consequence of Theorem 2.2 once we establish the fact that there is a 1–1 correspondence between the elements of $\Delta(B)$ and the nonzero Rudin equivalence classes determined by B as a subalgebra of $A(\Gamma)$. Rudin [11, p. 232] has shown that $\chi_E \in B$ for every nonzero Rudin equivalence class $E \subset \Gamma$. So for $h \in \Delta(B)$, let $\lambda = h(\chi_E)$. Then $\lambda = h(\chi_E) = h(\chi_E{}^2) = h(\chi_E)\, h(\chi_E) = \lambda^2$. Thus either $\lambda = 0$ or $\lambda = 1$. But since B is spanned by its idempotents, there is at least one nonzero Rudin equivalence class, E_h, for which $h(\chi_{E_h}) = 1$ (otherwise h would be 0 and therefore $h \notin \Delta(B)$). Now suppose $h(\chi_{E_j}) = 1$ for two distinct nonzero

Rudin equivalence classes E_1 and E_2. Then $0 = h(0) = h(\chi_{E_1} \cdot \chi_{E_2}) = h(\chi_{E_1}) h(\chi_{E_2}) = 1 \cdot 1 = 1$; contradiction. Thus for every $h \in \Delta(B)$ there corresponds a unique nonzero Rudin equivalence class E_h for which $h(\chi_{E_h}) = 1$. Now let E be an arbitrary nonzero Rudin equivalence class. Define the linear functional h on B by $h(g) = g(E)$ for each $g \in B$. Then $h \in \Delta(B)$ and $h(\chi_E) = \chi_E(E) = 1$; thus $E = E_h$. Hence there is a 1–1 correspondence between the nonzero Rudin equivalence classes and the elements of $\Delta(B)$. This completes the proof.

If the subalgebra, B, is not spanned by its idempotents, Corollary 2.4 need not hold. For example, Rider [10] has found a subalgebra B of $A(Z)$ which is not spanned by its idempotents and an $f \in B$ satisfying (a) and (b) of Corollary 2.4 for which $\{f^n\}_{n=1}^{\infty}$ is complete in B and for which $\{f^n\}_{n=2}^{\infty}$ is not complete in B. But Kahane [3] and Friedberg [7] have found some structural conditions on the Rudin equivalence classes which insure that B is spanned by its idempotents for the cases $\Gamma = Z$ and $\Gamma = Z \times Z$.

III. MÜNTZ'S THEOREM IN $A(T)$

The solution of the Müntz problem treated in this section has a close connection with problems treated in chapter six of Kahane's recent book [4].

DEFINITION 3.1. Let $f = \sum_{-\infty}^{\infty} a_n e^{in\theta} \in A(T)$. The entropy, $H(f)$, of f is defined as $H(f) = -\sum_{-\infty}^{\infty} | a_n | \log | a_n |$.

Remark 3.2. When $a_n \geqslant 0$ and $\sum_{-\infty}^{\infty} a_n = 1$, $H(f)$ is the entropy of the probability distribution on Z which assigns probability a_n to n; see Khinchin [6] and Mureika [8]. Note that $H(f)$ is not necessarily finite. For example if

$$\begin{aligned} a_n &= \frac{1}{| n | (\log | n |)^2} \qquad \text{for} \quad | n | \geqslant 2 \\ &= 0 \qquad\qquad\qquad\;\; \text{for} \quad | n | < 2, \end{aligned}$$

then $f \in A(T)$ and $H(f) = \infty$.

LEMMA 3.3. *$h(x) = -x \log x$ is monotonic nondecreasing on* $[0, 1/e]$.

Proof. $h'(x) = -x\, 1/x - \log x$; so $h'(x) \geqslant 0$ if $-1 - \log x \geqslant 0$, i.e., if $x \leqslant 1/e$.

LEMMA 3.4. *Let*

$$f = \sum_{-\infty}^{\infty} a_j e^{ij\theta} \in A(T), \qquad f_k = \sum_{-\infty}^{\infty} a_j^{(k)} e^{ij\theta} \in A(T), \qquad g = \sum_{-\infty}^{\infty} b_j e^{ij\theta} \in A(T).$$

(a) *$H(cf) = -|c|\log|c|\,\|f\| + |c|\,H(f)$.*

(b) *If $|a_j| + |b_j| \leqslant 1/e$ for all j, then $H(f+g) \leqslant H(f) + H(g)$.*

(c) *If $0 \leqslant a_j, b_j$ and $\|f\| \leqslant 1$, $\|g\| \leqslant 1$, then $H(fg) \leqslant \|g\|\,H(f) + \|f\|\,H(g)$.*

(d) *If $0 \leqslant a_j$ for all j, and $\|f\| \leqslant 1$, then $H(f^n) \leqslant n\,\|f\|^{n-1}\,H(f)$.*

(e) *If $H(f) < \infty$ and $H(g) < \infty$, then $H(f+g) < \infty$.*

(f) *If $\sum_k |a_j^{(k)}| \leqslant 1/e$ and $f = \sum_1^\infty f_k$ converges in $L^1(T)$, then $H(f) \leqslant \sum H(f_k)$.*

(g) *If $|a_j| \leqslant b_j$ and $\|g\| \leqslant 1/e$, then $H(f^n) \leqslant H(g^n)$, $n = 1, 2, 3, \ldots$.*

(h) *If $H(f) < \infty$ and $H(g) < \infty$, then $H(fg) < \infty$.*

Proof.

(a)
$$H(cf) = -\sum_{-\infty}^{\infty} |ca_n| \log |ca_n| = -\sum_{-\infty}^{\infty} |ca_n|\,[\log|c| + \log|a_n|]$$
$$= -|c|\log|c|\,\|f\| + |c|\,H(f).$$

(b)
$$H(f+g) = -\sum_{-\infty}^{\infty} |a_j + b_j| \log |a_j + b_j|$$
$$\leqslant -\sum_{-\infty}^{\infty} (|a_j| + |b_j|) \log(|a_j| + |b_j|) \text{ by Lemma 3.3}$$
$$\text{since } |a_j + b_j| \leqslant |a_j| + |b_j| \leqslant \frac{1}{e}$$
$$\leqslant -\sum_{-\infty}^{\infty} |a_j| \log |a_j| - \sum_{-\infty}^{\infty} |b_j| \log |b_j|$$
$$\text{since } |a_j| + |b_j| < 1$$
$$= H(f) + H(g).$$

(c)
$$H(fg) = -\sum_{n=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} a_{n-j}b_j\right) \log\left(\sum_{k=-\infty}^{\infty} a_{n-k}b_k\right)$$
$$\leqslant -\sum_n \sum_j a_{n-j}b_j \log(a_{n-j}b_j)$$
$$\text{since } a_{n-j}b_j \leqslant \sum_{k=-\infty}^{\infty} a_{n-k}b_k \leqslant 1$$
$$= -\sum_n \sum_j b_j a_{n-j} \log a_{n-j} - \sum_n \sum_j a_{n-j}b_j \log b_j$$
$$= -\sum_j b_j \sum_n a_{n-j} \log a_{n-j} - \sum_j b_j \log b_j \sum_n a_{n-j}$$
$$= \|g\|\,H(f) + \|f\|\,H(g).$$

(d) By (c) we have $H(f^2) \leqslant 2\,\|f\|\,H(f)$.

Proceed by induction and assume true for n: Then

$$H(f^{n+1}) \leqslant \|f^n\| \, H(f) + H(f^n) \, \|f\| \quad \text{by (c)}$$
$$\leqslant \|f\|^n \, H(f) + n \, \|f\|^{n-1} \, H(f) \, \|f\|$$

by the induction hypothesis

$$= (n+1) \, \|f\|^n \, H(f).$$

(e) $$H(f+g) = -\sum_{-\infty}^{\infty} |a_j + b_j| \log |a_j + b_j|$$
$$= -\sum_{-N}^{N} |a_j + b_j| \log |a_j + b_j|$$
$$-\sum_{|j|>N} |a_j + b_j| \log |a_j + b_j|.$$

It suffices to show that the second sum converges for some choice of $N < \infty$, since the first sum is always finite. Choose N such that $|a_j| + |b_j| \leqslant 1/e$ for $j > N$. Then by (b) the second sum converges since $H(f)$ and $H(g) < \infty$.

(f) Clearly $a_j = \sum_{k=1}^{\infty} a_j^{(k)}$ whenever $\sum_1^N f_k$ converges to f in $L^1(T)$ as $N \to \infty$. [In particular, this is true if $\sum f_k$ converges to f in $A(T)$.] Then

$$H(f) = -\sum_j \Big| \sum_k a_j^{(k)} \Big| \log \Big| \sum_k a_j^{(k)} \Big|$$
$$\leqslant -\sum_j \Big(\sum_k |a_j^{(k)}| \Big) \log \sum |a_j^{(k)}| \qquad \text{by Lemma 3.3}$$
$$\text{since} \quad \Big| \sum_k a_j^{(k)} \Big| \leqslant \sum_k |a_j^{(k)}| \leqslant \frac{1}{e}$$
$$\leqslant -\sum_j \sum_k |a_j^{(k)}| \log |a_j^{(k)}|$$
$$= -\sum_k \sum_j |a_j^{(k)}| \log |a_j^{(k)}|$$
$$= \sum_k H(f_k).$$

(g) $$\Big| \sum_j a_{k-j} a_j \Big| \leqslant \sum_j |a_{k-j}| \, |a_j| \leqslant \sum_j b_{k-j} b_j \, .$$

Thus if $f^n = \sum_j a_j^{(n)} e^{ij\theta}$ and $g^n = \sum_j b_j^{(n)} e^{ij\theta}$, it is clear that $|a_k^{(n)}| \leqslant b_k^{(n)}$ for $n = 1, 2$, and for $k \in Z$. Claim this is true for all n. Proceed by induction and assume true for n:

$$\begin{aligned} |a_k^{(n+1)}| &= \left| \sum_j a_{k-j}^{(n)} a_j \right| \leqslant \sum_j |a_{k-j}^{(n)}| \, |a_j| \\ &\leqslant \sum_j b_{k-j}^{(n)} b_j \qquad \text{by the induction hypothesis} \\ &= b_k^{(n+1)}. \end{aligned}$$

Thus $|a_k^{(n)}| \leqslant b_k^{(n)}$ for $n = 1, 2, 3,...$ and for $k \in Z$. Now clearly $b_k^{(n)} \leqslant 1/e$ for $n = 1, 2, 3,...$, and for $k \in Z$ since $\| g \| \leqslant 1/e$.

Thus $-|a_k^{(n)}| \log |a_k^{(n)}| \leqslant -b_k^{(n)} \log b_k^{(n)}$, $n = 1, 2, 3,...$ and $k \in Z$ by Lemma 3.3.

Therefore $H(f^n) \leqslant H(g^n)$, $n = 1, 2, 3,...$.

(h) Choose N such that

$$\sum_{|j|>N} (|a_j| + |b_j|) \leqslant \frac{1}{e},$$

and let

$$f_1 = \sum_{|j|<N} a_j e^{ij\theta}, \qquad g_1 = \sum_{|j|<N} b_j e^{ij\theta}.$$

Then

$$\begin{aligned} H(fg) &= H[[f_1 + (f - f_1)][g_1 + (g - g_1)]] \\ &= H[f_1 g_1 + (f - f_1) g_1 + f_1(g - g_1) + (f - f_1)(g - g_1)]. \end{aligned}$$

Now $H[f_1 g_1] < \infty$ since $f_1 g_1$ is a trigonometric polynomial.

$$H[(f - f_1) g_1] = H\left[\sum_{|j|<N} b_j e^{ij\theta}(f - f_1) \right] < \infty$$

by part (e) since

$$H[b_j e^{ij\theta}(f - f_1)] = H[b_j(f - f_1)] < \infty$$

by part (e) since $H[f] < \infty$ and $H[f_1] < \infty$. $H[f_1(g - g_1)] < \infty$ for the same reason. $H[(f - f_1)(g - g_1)] < \infty$ by part (g) (with $n = 1$) and by part (c). Thus, by part (e), $H[fg] < \infty$.

LEMMA 3.5. *If $f \in A(T)$ and $f'' \in A(T)$ then $H(f) < \infty$.*

Proof.

$$\begin{aligned} H(f) &= -\,|\,a_0\,|\log|\,a_0\,| - {\sum}'\,|\,a_n\,|\log|\,a_n\,| \\ &= -\,|\,a_0\,|\log|\,a_0\,| - {\sum}'\,|\,a_n\,|^{1/2}\,|\,a_n\,|^{1/2}\,|\,n\,|\,\frac{1}{|\,n\,|}\log|\,a_n\,| \\ &\leqslant -\,|\,a_0\,|\log|\,a_0\,| + \left[{\sum}'\,(|\,a_n\,|^{1/2}\,|\,n\,|)^2\right]^{1/2} \\ &\qquad \times \left[{\sum}'\left(\frac{1}{|\,n\,|}\,|\,a_n\,|^{1/2}\log|\,a_n\,|\right)^2\right]^{1/2} \end{aligned}$$

by Schwarz's inequality

$$\leqslant -\,|\,a_0\,|\log|\,a_0\,| + C\,\|\,f''\,\|^{1/2} < \infty.$$

(Note that $\{|\,a_n\,|\,(\log|\,a_n\,|)^2\}$ is a bounded sequence because

$$\lim_{x\to 0} x(\log x)^2 = \lim_{x\to 0}\frac{2(\log x)\,1/x}{-1/x^2} = \lim_{x\to 0} -\,2x\log x = 0.)$$

LEMMA 3.6. *Suppose $f\in A(T)$ and $H(f)<\infty$ and $F(z)=\sum_0^\infty a_n(z-c)^n$ converges for $|\,z-c\,|<r$ and $\|f-c\|=r_1<\min(1/e,r)$. Then $H[F(f)]<\infty$.*

Proof. Clearly $F(f)=\sum_0^\infty a_n(f-c)^n$ converges in $A(T)$. Choose $N>1$ such that

$$\sum_{N+1}^{\infty}|\,a_n\,|\,r_1{}^n < \frac{1}{e}.$$

Then

$$H[F(f)] = H\left[\sum_0^N a_n(f-c)^n + \sum_{N+1}^{\infty} a_n(f-c)^n\right] < \infty$$

if $H[\sum_{N+1}^\infty a_n(f-c)^n]<\infty$, by Lemma 3.4(e) and (h). Now by Lemma 3.4(f),

$$\begin{aligned} H\left[\sum_{N+1}^{\infty} a_n(f-c)^n\right] &\leqslant \sum_{N+1}^{\infty} H[a_n(f-c)^n] \\ &= \sum_{N+1}^{\infty}\{-\,|\,a_n\,|\log|\,a_n\,|\,\|(f-c)^n\| + |\,a_n\,|\,H[(f-c)^n]\} \end{aligned}$$

by Lemma 3.4(a)

$$\leqslant \sum_{N+1}^{\infty}\{-\,|\,a_n\,|\log|\,a_n\,|\,\|f-c\|^n + |\,a_n\,|\,n\,\|f-c\|^{n-1}H(f-c)\}$$

by Lemma 3.4(g) and (d)

$$\leqslant \sum_{N+1}^{\infty} (-\mid a_n \mid \log \mid a_n \mid) r_1^n + H(f-c) \sum_{N+1}^{\infty} n \mid a_n \mid r_1^{n-1}.$$

The second series obviously converges, since $r_1 < r$. The first series converges since $-\sum \mid a_n \mid r_1^n \log(\mid a_n \mid r_1^n) < \infty$ by Lemma 3.5. Hence $H[\sum_{N+1}^{\infty} a_n(f-c)^n] < \infty$.

THEOREM 3.7. *Suppose $f \in A(T)$ and $f(\theta) \nleqslant 0$ on T and $H(f) < \infty$. Then $H(\log f) < \infty$.*

Proof. By Wiener's theorem $1/f \in A$. Therefore we can find a trigonometric polynomial, g, such that $\| 1 - gf \| < 1/e$ and such that $g(\theta) \nleqslant 0$. (e.g., we may take g as $S_N[1/f]$ for a sufficiently large N). Also

$$H(\log f) = H(\log gf + \log 1/g) < \infty \quad \text{if} \quad \begin{cases} H(\log gf) < \infty \\ H(\log 1/g) < \infty \end{cases}$$

by Lemma 3.3(e). Since $\log(1/g) \in C^\infty(T)$, by Lemma 3.5, $H(\log 1/g) < \infty$. By Lemma 3.6, letting

$$F(z) = \log z = \sum_{1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n},$$

we see that $H(\log gf) < \infty$. Thus $H(\log f) < \infty$.

THEOREM 3.8. *Suppose $f \in A(T)$ and $f(\theta) \neq 0$ on T and $H(f) < \infty$. Then $H(1/f) < \infty$.*

Proof. Virtually the same as Theorem 3.7.

Theorems 3.7 and 3.8 make the following conjecture worth investigating:

Conjecture 3.9. If $F(z)$ is analytic on a domain D and if $f \in A(T)$ and Range $f \subset D$ and $H(f) < \infty$ then $H[F(f)] < \infty$.

But the proofs of Theorems 3.7 and 3.8 use *special* properties of the functions $\log z$ and $1/z$, respectively, which apparently preclude a direct adaptation of these proofs to the case of a more general $F(z)$.

For our Müntz theorem on $A(T)$ we need the following result (Boas [1], p. 156) concerning the zeros of an analytic function.

THEOREM 3.10. *Suppose $F(z)$ is analytic and of exponential type for $\operatorname{Re} z \geqslant 0$ and that $\{\mu_n\}_1^\infty$ is an infinite sequence of distinct positive numbers. Also suppose*

(a) $F(\mu_n) = 0 \qquad n = 1, 2, 3, \ldots$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\mu_n} = \infty$

(c) $\displaystyle\varlimsup_{R \to \infty} \int_1^R \frac{1}{y^2} \log | F(iy) F(-iy)| \, dy < \infty.$

Then $F(z) \equiv 0$ *in* $\operatorname{Re} z \geqslant 0$.

LEMMA 3.11. *Suppose*

$$g = \sum_{-\infty}^{\infty} b_n e^{in\theta} \in A(T) \cap C'(T).$$

Then $\| g \|_{A(T)} \leqslant K \| g' \|_{L^2(T)} + | b_0 |$ (*K independent of* g).

Proof.

$$\| g \|_{A(T)} = \sum_{-\infty}^{\infty}{}' n \, | b_n | \frac{1}{n} + | b_0 |$$

$$\leqslant \| g' \|_{L^2(T)} \left[\sum_{-\infty}^{\infty}{}' \frac{1}{n^2} \right]^{1/2} + | b_0 |$$

$$= K \| g' \|_{L^2(T)} + | b_0 |.$$

LEMMA 3.12. *For y real,* $\| e^{iy \cos \theta} \|_{A(T)} \leqslant 1 + K | y |$ *and* $\| e^{iy \sin \theta} \|_{A(T)} \leqslant 1 + K | y |$.

Proof. By Lemma 3.11,

$$\| e^{iy \cos \theta} \|_{A(T)} \leqslant \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iy \cos \theta} \, d\theta \right| + K \| iy \, e^{iy \cos \theta} \sin \theta \|_{L^2(T)}$$

$$\leqslant 1 + K | y |.$$

Similarly,

$$\| e^{iy \sin \theta} \|_{A(T)} \leqslant \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iy \sin \theta} \, d\theta \right| + K \| iy \, e^{iy \sin \theta} \cos \theta \|_{L^2(T)}$$

$$\leqslant 1 + K | y |.$$

LEMMA 3.13. *Suppose* $g \in A(T)$ *and* λ *is an integer; let* $g_1(\theta) = g(\lambda\theta)$. *Then* $\| g_1 \|_{A(T)} = \| g \|_{A(T)}$.

Proof. Let $g = \sum_{-\infty}^{\infty} b_n e^{in\theta}$.

$$\| g_1 \| = \Big\| \sum_{n \in Z} b_n e^{i\lambda n\theta} \Big\| = \sum_{n \in Z} | b_n | \qquad \text{since } \lambda n \in Z \text{ for each } n$$
$$= \| g \|.$$

THEOREM 3.14. *Suppose $f = \sum_{-\infty}^{\infty} a_n e^{in\theta} \in A(T)$ and $f > 0$ and $H(f) = -\sum | a_n | \log | a_n | < \infty$. Let B be the closed subalgebra of $A(T)$ generated by f. Let $\{\mu_n\}_1^\infty$ be an infinite sequence of distinct positive numbers, and suppose $\sum (1/\mu_n) = \infty$. Then $\{f^{\mu_n}\}_1^\infty$ is complete in B.*

Proof. Suppose $\log f(\theta) = g(\theta) = \sum_{-\infty}^{\infty} b_n e^{in\theta}$; then $g \in A(T)$. Suppose ϕ is a continuous linear functional on $A(T)$ for which $\langle \phi, f^{\mu_n} \rangle = 0$, $n = 1, 2, 3, \ldots$. We must show that $\langle \phi, f^j \rangle = 0$, $j = 1, 2, 3, \ldots$. Define $F(z) = \langle \phi, e^{zg} \rangle$. Then $F(z)$ is an entire function of exponential type with zeros at $\{\mu_n\}_1^\infty$ on the positive real axis. We have for real y

$$| F(iy) F(-iy) | \leqslant \| \phi \|^2 \| e^{iyg} \| \| e^{-iyg} \|.$$

Now $g(\theta) = \sum_{-\infty}^{\infty} b_n e^{in\theta} = \sum_0^\infty c_n \cos n\theta + \sum_1^\infty d_n \sin n\theta$, where $c_0 = b_0$ and

$$\left.\begin{aligned} c_n &= 2 \operatorname{Re} b_n \\ d_n &= 2 \operatorname{Im} b_n \end{aligned}\right\} \qquad \text{for} \quad n \geqslant 1.$$

Thus

$$\begin{aligned} \| e^{iyg} \| &= \| e^{iy[\sum_0^\infty c_n \cos n\theta + \sum_1^\infty d_n \sin n\theta]} \| \\ &= \Big\| \prod_1^\infty e^{iyc_n \cos n\theta} \prod_1^\infty e^{iyd_n \sin n\theta} \Big\| \\ &\leqslant \prod_1^\infty \{ \| e^{iyc_n \cos n\theta} \| \| e^{iyd_n \sin n\theta} \| \} \\ &= \prod_1^\infty \{ \| e^{iyc_n \cos \theta} \| \| e^{iyd_n \sin \theta} \| \} \qquad \text{by Lemma 3.13} \\ &\leqslant \prod_1^\infty [1 + K | y | | c_n |][1 + K | y | | d_n |] \quad \text{by Lemma 3.12} \\ &\leqslant \prod_1^\infty [1 + 2K | y | | b_n |]^2. \end{aligned}$$

Thus

$$| F(iy) F(-iy) | \leqslant \| \phi \|^2 \left\{ \prod_1^\infty [1 + 2K | y | | b_n | \right\}^4.$$

Therefore

$$\int_1^\infty \frac{1}{y^2} \log |F(iy)F(-iy)|\, dy$$

$$\leqslant \int_1^\infty \frac{1}{y^2} \log \left\{\| \phi \|^2 \prod_1^\infty [1 + 2Ky \mid b_n \mid]^4\right\} dy$$

$$= \int_1^\infty \frac{2 \log \| \phi \|}{y^2}\, dy + \int_1^\infty \sum_1^\infty \frac{4 \log[1 + 2K \mid b_n \mid y]}{y^2}\, dy$$

$$= 2 \log \| \phi \| \int_1^\infty \frac{1}{y^2}\, dy + 4 \sum_1^\infty \int_1^\infty \frac{\log[1 + 2K \mid b_n \mid y]}{y^2}\, dy.$$

Now

$$\int_1^\infty \frac{1}{y^2} \log[1 + K \mid b_n \mid y]\, dy$$

$$= \log[1 + K \mid b_n \mid y] \left[-\frac{1}{y}\right]\Big|_1^\infty + \int_1^\infty \frac{1}{y} \frac{K \mid b_n \mid}{1 + K \mid b_n \mid y}\, dy$$

$$= \log[1 + K \mid b_n \mid] + \int_1^\infty \frac{1}{y} \frac{K \mid b_n \mid}{1 + K \mid b_n \mid y}\, dy.$$

Consider the second term:

$$\int_1^R \frac{1}{y} \frac{K \mid b_n \mid}{1 + K \mid b_n \mid y}\, dy$$

$$= \int_1^R \left[\frac{K \mid b_n \mid}{y} - \frac{K^2 \mid b_n \mid^2}{1 + K \mid b_n \mid y}\right] dy$$

$$= K \mid b_n \mid \log R - \frac{K^2 \mid b_n \mid^2}{K \mid b_n \mid} \int_{1+Kb_n}^{1+RKb_n} \frac{1}{w}\, dw$$

$$= K \mid b_n \mid \{\log R - \log[1 + RK \mid b_n \mid] + \log[1 + K \mid b_n \mid]\}$$

$$= K \mid b_n \mid \log \frac{R[1 + K \mid b_n \mid]}{1 + RK \mid b_n \mid} \to K \mid b_n \mid \log \frac{1 + K \mid b_n \mid}{K \mid b_n \mid} \quad \text{as} \quad R \to \infty.$$

Thus, putting things together:

$$\sum_1^\infty \int_1^\infty \frac{1}{y^2} \log[1 + K \mid b_n \mid y]\, dy$$

$$= \sum_1^\infty \left\{\log[1 + K \mid b_n \mid] + K \mid b_n \mid \log \frac{1 + K \mid b_n \mid}{K \mid b_n \mid}\right\}$$

$$\leqslant \sum_1^\infty K \mid b_n \mid + \sum_1^\infty K \mid b_n \mid \log[1 + K \mid b_n \mid] - \sum_1^\infty K \mid b_n \mid \log (K \mid b_n \mid)$$

$$\leqslant K \sum_1^\infty \mid b_n \mid + \sum_1^\infty K^2 \mid b_n \mid^2 - K \log K \sum_1^\infty \mid b_n \mid - K \sum_1^\infty \mid b_n \mid \log \mid b_n \mid$$

$$\leqslant [K - K \log K] \| g \| + K^2 \| g \|^2 + KH(g).$$

Now we know that $g = \log f \in A(T)$ (because $f > 0$ and hence $\log z$ is analytic on the range of f). Thus $\| g \| < \infty$. Also, $H(g) < \infty$ by Theorem 3.7. Therefore we have

$$\int_1^\infty \frac{1}{y^2} \log | F(iy) F(-iy)| \, dy < \infty.$$

Therefore, Theorem 3.10 applies and so $F(z) \equiv 0$. In particular, $F(j) = \langle \phi, f^j \rangle = 0$ for every positive integer j.

References

1. R. P. Boas, Jr., "Entire Functions," Academic Press, New York, 1954.
2. P. J. Davis, "Interpolation and Approximation," Blaisdell, Waltham, Mass., 1963.
3. J. P. Kahane, Idempotents and closed subalgebras of $A(Z)$, "Function Algebras," Proceedings of the International Symposium on Function Algebras, Tulane University, 1965, pp. 198–207, Scott, Foresman and Company, Chicago, Ill., 1966.
4. J. P. Kahane, "Series de Fourier Absolument Convergentes," Springer–Verlag, Berlin, Germany, 1970.
5. Y. Katznelson and W. Rudin, The Stone–Weierstrass property in Banach algebras, *Pacific J. Math.* **11** (1961), 253–265.
6. A. I. Khinchin, "Mathematical Foundations of Information Theory," Dover Publications, Inc., New York, 1957.
7. S. Friedberg, Closed subalgebras of group algebras, *Trans. Am. Math. Soc.* **147** (1970), 117–125.
8. R. Mureika, The maximization of entropy of discrete denumerably valued random variables with known mean, *Ann. Math. Stat.* **43** (1972), 541–552.
9. D. J. Newman, E. Passow, and L. Raymon, Approximation by Müntz Polynomials on Sequences, *J. Approximation Theory* **1** (1968), 476–483.
10. D. Rider, Closed subalgebras of $L^1(T)$, *Duke Math. J.* **36** (1969), 105–115.
11. W. Rudin, "Fourier Analysis on Groups," Interscience, New York, 1962.